

General Aspects of Symmetry Breaking

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Abstract

On basis of an algebraic analysis of symmetry breaking in general and the Higgs mechanism in the standard model of elementary particles we generalize the concept of symmetry breaking to systems with non-compact groups but not necessarily caused by a potential. Thereto we give some simple, but unfamiliar examples of symmetry breaking with and without potentials. The analysis of the concept of mass in space-time and in the Higgs mechanism will lead to a model unifying both structures in terms of symmetry breaking of $GL(2, \mathbb{C})$.

Introduction

In his famous work of 1916 [1] A. Einstein motivated the necessity of an extension of special relativity to general relativity with the help of a gedanken-experiment based on Mach's principle. It states that it is not possible to experience the motion of a single isolated object in space-time without reference to other objects. Einstein concluded that the absolute Galileian or Newtonian space-time, and equally the concept of space-time in special relativity, is just an "imaginary cause" (fiktive Ursache [1]). His postulate of general covariance states that there is no given absolute basis in space-time, i.e. no given reference frame, and therewith no given form of the metric, but the local bases and therewith the local metrical structure is due to the mass distribution in space-time.

Pushing Mach's principle to the extreme we have to state that we can't experience space-time without any objects to test space-time (also fields, like the electro-magnetic field, have to be tested with the help of objects), and we can't define length or volume, mass or energy and (angular) momentum without reference objects. On the other hand objects themselves are always regarded as objects in space-time. Hence this serves for a paradox.

Where does our concept of a particle (object) and its mass come from? Firstly, the definition of a particle in special relativity is given by Wigner [2], particles being positive unitary irreducible representations of the Poincaré group. Therewith mass is the Casimir invariant of the translations within the Poincaré group and therewith the abstract concept of inertial mass. Modern quantum field theory, especially the standard model of elementary particles [3], has given another approach to the concept of mass, connected with the Higgs mechanism [4]. The Higgs mechanism creates in a gauge invariant way at least formally terms within the Lagrange densities, which are already known to be mass terms for particles, leading to already known equation of motions for massive particles. There is no conceptual theory why these 'masses', generated out of an internal symmetry structure, are equal to the external concept of mass according to the Poincaré group. Even more there is no conceptual connection to the role of mass in general relativity.

The mass terms in the standard model are generated by the concept of spontaneous symmetry breaking. With the above problems in mind we analyze in this work the general structures of symmetry breaking. This is done with the aim to get rid of the existence of an a priori given potential, which already presupposes the concept of energy. Because of the

non-compact structure of the external space-time symmetry this analysis is also performed to generalize the concept of symmetry breaking of compact groups to non-compact groups. Based on this analysis, specified by some simple examples, we propose a quite abstract model which has the potential to resolve the paradox and explain the duality of space-time and objects in terms of symmetry breaking.

Algebraic Aspects of Symmetry Breaking

Spontaneous symmetry breaking is an important feature of the standard model of elementary particles as well as for grand unified theories. Besides other features it is essential to define the observed symmetry patterns and to generate in a gauge invariant way mass terms for gauge bosons and fermions. To name the basic aspects of symmetry breaking the Higgs sector of the standard model will be analyzed shortly: Omitting the kinetic terms, the Higgs sector of the standard model is given by an $U(2)$ -symmetric potential on a complex 2-dimensional vector space V of the Higgs field $\Phi(x)$:

$$\Phi(x) \in V \cong \mathcal{C}^2, \quad x \in M$$

$$V_{\text{Higgs}}(\Phi) = \mu^2 \Phi^* \Phi + \lambda (\Phi^* \Phi)^2, \quad \lambda > 0. \quad (1)$$

For $\mu^2 > 0$ no symmetry breaking occurs, i.e. the ground state (state of lowest potential energy) is unique and invariant under the action of the whole symmetry group $U(2)$. For $\mu^2 < 0$ the possible ground states are only invariant under the action of a $U(1)$ subgroup of $U(2)$. The manifold of all possible ground states is given by the 3-parametric (Goldstone) coset space $U(2)/U(1)$. This situation is often exemplified in \mathbb{R}^2 . Take the \mathbb{R}^2 -dimensional analogue potential with $O(2)$ -symmetry:

$$y \in \mathbb{R}^2$$

$$V(y) = ay^2 + b(y^2)^2, \quad b > 0. \quad (2)$$

For $a > 0$ the ground state is unique, given by the origin, with complete symmetry $O(2)$. For $a < 0$ (Mexican hat potential) there is a S^1 -sphere (circle) of ground states defined by the vectors with length $|y| = \sqrt{\frac{-a}{2b}} =: R$.

Contrary to the Higgs potential the invariance groups of these vectors are discrete. The manifold of ground states is given with $O(2)/\mathbb{I}_2 \cong SO(2) \cong \mathbb{S}^1$ (with $\mathbb{I}_2 \cong \{\mathbb{I}_2, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}$ in a certain basis).

To distinguish between $a > 0$ and $a < 0$ is like to distinguish between the different strata of the action of $O(2)$ on \mathbb{R}^2 . A stratum is the collection of all isomorphic orbits of the action of a group. In general: The action of a group G on a vector v of a representation space V of this group defines the orbit $[v]_G$ of this vector with respect to the representation ρ of the group:

$$\begin{aligned} \text{representation : } \rho &: G \longrightarrow \text{end}(V); \\ \text{endomorphism : } \rho(g) &: V \longrightarrow V, \quad g \in G; \\ G\text{-action : } \rho(g)(v) &=: g \bullet v \in V; \\ \text{orbit : } [v]_G &:= \rho(G)(v) = G \bullet v. \end{aligned}$$

V is thus decomposed non-linearly into orbits of the action of G . There may be a non-trivial subgroup $H_v \subset G$ which leaves the vector v invariant, i.e.

$$H_v \bullet v = v.$$

This subgroup is called fixgroup or isotropy group of v or, due to Wigner [2], little group. Regarded as manifold the orbit $[v]_G$ is isomorphic to the coset (or homogeneous) space G/H_v ,

$$[v]_G \cong G/H_v.$$

If H_v is a normal subgroup this manifold has in addition a group structure. If v and w are two different elements of the same orbit there is at least one $g \in G$ with $w = g \bullet v$. Therewith v and w have isomorphic but in general not the same fixgroups:

$$H_w = gH_v g^{-1}.$$

I.e. the fixgroups of elements of one orbit are conjugated. Thus the concept of a little group is a ‘local’ structure on the manifold G/H : its representation depends on the element of the manifold. The collection of all orbits with isomorphic fixgroups, i.e. the same fixgroup up to conjugation, is called a stratum. Thus V is decomposed non-linearly into strata with smaller decomposition given by the orbits.

With this notation the two examples for spontaneous symmetry breaking given above can be characterized as follows: A G -symmetric potential defines via its ground states one orbit within a stratum. In case of the little group

of this orbit being isomorphic to the whole group $H \cong G$ the orbit is trivial, $[v]_G \cong G/G \cong \{1\}$, i.e. the ground state is unique. Otherwise we have a non-trivial orbit and therewith a non-trivial manifold of ground states. In the example of $O(2)$ acting on \mathbb{R}^2 the non-trivial stratum is the collection of all concentric circles around the origin. These orbits within the stratum are connected with a dilatation operation, i.e. by the action of the group $D(1) = e^{\mathbb{R}} \cong \mathbb{R}^+$. For the non-trivial stratum the selection of an orbit within the stratum introduces a length scale, $|y| = R$ (which in the Higgs model becomes an energy or mass scale), and therefore breaks the $D(1)$ invariance of the stratum. I.e. hidden in the potential there is an explicit breaking of the $D(1)$ structure of the stratum by an explicit introduction of a scale, e.g. $\sqrt{\frac{-a}{2b}}$ or $\sqrt{\frac{-\mu^2}{2\lambda}}$. Spontaneous symmetry breaking now distinguishes one vector within the orbit. For all of these vectors the action of the group G on this vector generates (per definition) the orbit. The action of the group together with the action of $D(1)$ generates the whole stratum out of one vector.

In this context spontaneous symmetry breaking has two features: First fixing a scale within the stratum and secondly distinguishing one vector which has the possibility to give one basis vector for the representation space. Together this will provide one normal basis vector of the representation space of G . Notice, within a vector space there is normally given no natural, i.e. preferred, basis. However, mostly we are used to work in a basis. On the other side general relativity teaches to understand nature basis-free as long as nature itself doesn't distinguish a basis. It is the distinction of one preferred (basis) vector which causes symmetry breaking. To ask for symmetry breaking is first of all to ask for the distinction of a non-trivial vector within a representation space of a symmetry group. Therefore the description given above focuses on an algebraic and therewith basis-free description of symmetry breaking. We will stress this point of view in the examples given below.

Of course the algebraic description of symmetry breaking in terms of orbits or cosets and little groups is well known, see e.g. [5]. However, these structures are only used in connection with compact symmetry groups G due to the assumption of the existence of potentials like (1) or (2). Here we emphasized certain structures that will be generalized to systems with broken symmetries with and without potentials. Even more symmetry breaking of non-compact groups will be considered.

Simple Examples for Symmetry Breaking in General

There are many examples known in physics which cause spontaneous symmetry breaking by potentials of the form (1) or (2), e.g. ferromagnetism, superconducting and superfluidity. We like to give some purely structural examples of symmetry breaking in a more general sense. Hence, due to the missing of dynamics, no massless Goldstone excitations will appear. However these examples are examples of symmetry breaking in general, mostly not familiar in connection with symmetry breaking, but rather trivial. Not in all cases a potential is needed. The same effect of breaking a global symmetry may equally be caused by initial conditions in a dynamical problem. In both structures, spontaneous symmetry breaking caused by the ground state of a potential and symmetry breaking by the initial conditions, the basic ingredient is the distinction of a non-trivial vector in the representation space of a group.

In the following we like to give some really simple examples to make clear the basic concepts of symmetry breaking within our familiar 3-space $\mathcal{S}_3 \cong \mathbb{R}^3$. The symmetry group of 3-space is the 3-dimensional Euclidean group $E(3) \cong O(3) \times_s \mathcal{S}_3$ (here \times_s denotes the semi-direct group product). The orbit of every vector within \mathcal{S}_3 generated by the action of $E(3)$ is the whole vector space \mathcal{S}_3 due to the affine normal subgroup \mathcal{S}_3 . However, \mathcal{S}_3 is decomposed in two different strata with respect to the action of $O(3)$. The trivial stratum is given by the origin of \mathcal{S}_3 and the non-trivial stratum includes every non-trivial vector. The little groups of the non-trivial vectors in \mathcal{S}_3 are isomorphic to $O(2)$. Therefore every non-trivial orbit is isomorphic to $O(3)/O(2)$ which itself is isomorphic to the 2-sphere¹ \mathbb{S}^2 . With this simple mathematical background one can imagine several examples of breaking $O(3)$ down to $O(2)$ whenever a non-trivial vector is given in 3-space.

Take a ball, which is the prototype of a $O(3)$ -symmetric object. When acting with an $O(3)$ -rotation on the ball nothing changes. However, when acting with a non-trivial translation $t \in \mathcal{S}_3$ onto the ball, i.e. the ball is moving, this movement t defines a non-trivial vector in \mathcal{S}_3 . The residual symmetry group of this system is the little group of t , isomorphic to $O(2)$. The direction of the movement can be characterized by a dot on the surface of the ball which would not change its location when acting with $O(2)$ onto the ball. All possible directions are given with all possible dots on the surface of the ball which is the 2-sphere $\mathbb{S}^2 \cong O(3)/O(2)$. Here the breaking of the

¹in general $O(n+1)/O(n) \cong \mathbb{S}^n$

symmetry is due to the initial conditions.

The breaking of $O(3)$ -symmetry of 3-space to $O(2)$ is even quite closer to us. Imagine the earth with its gravitational attraction. This defines a $O(3)$ -symmetric system. Now imagine you are going down to earth e.g. with a space ship. You have to set down on a single point on earth. However, now the $O(3)$ -symmetry is ‘lost’. Bound to the earth’s surface by the gravitational attraction everybody defines a non-trivial vector from the center to the surface of the earth. We individually have no freedom of the whole $O(3)$ -symmetry of 3-space, but feel only the residual $O(2)$ -symmetry of the tangent space onto the earth’s surface (e.g. when dancing on the floor), defined by the non-trivial vector from the center of the earth to ourself. However, when losing the global symmetry we gain the new, but local concepts of ‘up’² and ‘down’. Hence, symmetry breaking not only reduces the symmetry of the system but at the same time creates new structures by decomposing the representation space. This decompositions can locally be interpreted as vector space decompositions. In our example it is a decomposition of \mathcal{S}_3 into the tangential plane and the orthogonal line (up & down) due to the one distinguished (basis) vector³, $\mathcal{S}_3 \cong \mathbb{R} \oplus \mathcal{S}_2$, with $\mathcal{S}_2 \cong \mathbb{R}^2$. This decomposition is only given as local tangent space onto the global structure. The global structure however is a manifold decomposition of the orbits within the strata, i.e. $\mathcal{S}_3 \cong D(1) \times O(3)/O(2) \uplus \{0\}$. Therefore motion on the ‘Goldstone manifold’ $O(3)/O(2)$, i.e. on the earth’s surface, takes no (potential) energy, in analogue to the massless Goldstone excitations. On the other hand the $D(1)$ direction (up & down) is the analogue to the massive Higgs excitation⁴. This quite comprehensible example provides thus all properties of spontaneous symmetry breaking.

One remark to our local $O(2)$ -symmetry: How can we see the relevance of this residual symmetry for us individually? We are not $O(2)$ -symmetric, whereas plants, bound to a single point on earth, are (more or less) $O(2)$ -symmetric, best example may be given by a fir. For a tree there is normally no

²The absolute ‘up’ direction in say Australia and Europe may be quite different, however, no misunderstandings would arise.

³In general a complete basis of a vector space defines a complete decomposition of the vector space with respect to its field and therewith an isomorphism, e.g. $\mathcal{S}_3 \cong \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \cong \mathbb{R}^3$.

⁴Notice, that it is the gravitational potential which is the prototype for all other potentials and thus also for the Higgs potential. These analogies are therefore not astonishing but quite natural.

reason to prefer one direction. This is different for all kinds of moving objects on earth (biological or artificial). The possibility to change the location again breaks the $O(2)$ -symmetry down to discrete group of space reflections denoted by \mathbb{I}_2 . Hence, we and all other moving objects are (again more or less) symmetric under reflection of the space. Therefore we can distinguish our local surroundings in what's ahead and what's behind us. It is harder to define what's right and what's left.

Back to the example of symmetry breaking of every individual on earth. We are forced to live on the earth's surface by the $1/r$ gravitational potential of the earth on the one side and the resistance of the matter of the earth on the other side. Hence, everybody (and every body on earth) defines one different $O(2)$ -symmetric ground state in this $O(3)$ -symmetric potential. This is like spontaneous symmetry breaking in the Higgs mechanism. However, for the system earth there is also a breaking of the $O(3)$ -symmetry down to $O(2)$ given by dynamics, i.e. by the rotation of the earth itself, defining the north-south axis. Does there arise a problem when two different structures of symmetry breaking occur? Locally this seems to be not any problem. However, when moving on the earth's surface, e.g. from north to south, there arise Coriolis forces due to the rotation of the earth. These Coriolis forces vanish when travelling according to the symmetry of the rotation axis, i.e. according to the little group of the axial vector.

We have shown the most basic concepts connected with symmetry breaking in quite vivid examples. We can now take the road to more abstract systems in which symmetry breaking occurs.

Massive Objects

In the example given above symmetry breaking was considered in 3-space with the action of $O(3)$. This system is embedded in Minkowski space with the action of the Lorentz group $O(1, 3)$. In 1908 Hermann Minkowski began his famous talk in front of the 'Gesellschaft der Ärzte und Naturforscher' with the words: 'Von Stund an sollen Raum für sich und Zeit für sich völlig zu Schatten herabsinken und nur noch eine Art Union der beiden soll Selbständigkeit bewahren'⁵ [6]. Notice, when using the word 'from now on' Minkowski himself distinguished between time and space (he didn't say 'from

⁵From this hour on space itself and time itself should vanish into shadows but only a kind of union of both will regard independence.

now and here on'). Why are for us individually space and time distinguished? Again this could be a matter of a local vector space decomposition coming along with symmetry breaking. Like everybody on earth defines a non-trivial vector to the earth's surface, everybody, and every massive object, defines a massive vector p in Minkowski space ($p_\mu = (m, 0, 0, 0)$ in a certain basis). The little group of this vector is isomorphic to $O(3)$, compatible with the local vector space decomposition $M \cong \mathcal{T} \oplus \mathcal{S}_3$. Let us stress this again: Every massive object separates the action of the Lorentz group into the action of the space rotations $O(3)$, the little group acting trivial, and the Lorentz boosts $O(1, 3)/O(3)$. This provides a local decomposition of Minkowski space into time and space. Unfortunately, unlike in the above example, where we in principle can leave earth to feel the whole $O(3)$ -freedom, we can't get out of this decomposition to experience the whole Minkowski space, if there is any linear space-time, and to feel the whole freedom of the Lorentz group.

The embedding of this local or individual space-time decomposition $\mathcal{T} \oplus \mathcal{S}_3$ into Minkowski space M is given by special relativity. However, the global decomposition coming along with a massive vector is the manifold decomposition of the interior part of the forward lightcone, the stratum of the massive vector, into the Lorentz hyperboloids given with the manifold $D(1) \times SO^+(1, 3)/SO(3)$. Here again the non-trivial vector p_μ defines one orbit within the time-like stratum breaking the $D(1)$ structure of the stratum and introducing a scale. The tangent space on every point of this manifold is the Minkowski space in a specific space-time decomposition. This manifold can therefore be regarded as a prototype for a space-time manifold [7], i.e. locally Minkowskian, generated by a massive object. Apart from a potential, all structures of symmetry breaking are present in this example.

Our individual left-right-symmetry is thus the result of a pattern of symmetry breaking down from the Lorentz group:

- Being massive we define a local space-time decomposition in Minkowski space, $M \cong \mathcal{T} \oplus \mathcal{S}_3$, providing the concept of time itself and 'at the same time' the freedom of 3-space with its $O(3)$ -symmetry.
- Bound to earth we define our local flat surroundings with $O(2)$ -symmetry, the tangential plane, and the orthogonal direction, $\mathcal{S}_3 \cong \mathcal{S}_2 \oplus \mathcal{S}$. For us, being no birds, the vertical direction is quite different to the tangential plane like time is different to space.
- The possibility to move defines the concept of 'ahead' and 'behind',

depending on the direction of our (potential) motion, $\mathcal{S}_2 \cong \mathcal{S} \oplus \mathcal{S} \cong \mathbb{R} \oplus \mathbb{R}$, with residual reflection symmetry.

Evolution built our bodies (with some exceptions like the location of our heart) according to this symmetry structure. How could it have done different?

Wigner Classification in Terms of Symmetry Breaking

The example of a massive vector in Minkowski space is one part of the Wigner classification of particles [2] interpreted in terms of symmetry breaking. Particles are, according to Wigner, positive unitary irreducible representations of the Poincaré group. Therewith particles and their properties are classified due to the different $O(1,3)$ strata⁶ in Minkowski space with momentum vector p being

1. time-like,
2. light-like,
3. vanish,
4. space-like.

The 1st class - our previous example of massive particles - is characterized by real non-zero mass and discrete spin due to the spectrum of the compact little group $SO(3)$ for massive vector bosons (spin 1 with 3rd components $\pm 1, 0$) or its covering group $SU(2)$ for massive fermions (spin 1/2 with 3rd components $\pm 1/2$). Compatible with the action of this little group is the local space-time decomposition $\mathbb{M} \cong \mathbb{T} \oplus \mathcal{S}_3$ (or Sylvester decomposition).

The 2nd class, the stratum of zero-mass but non-vanishing vectors, is characterized by little groups isomorphic to $E(2)$. However, the vector space decomposition into one light-like direction \mathbb{L} is not compatible with the action of this little group, since the direct complement of \mathbb{L} is not invariant under the action of $E(2) \cong SO(2) \times_s \mathbb{R}^2$. I.e. $E(2)$ acts not irreducible on this decomposition, the action of $\mathbb{R}^2 \subset E(2)$ leaves invariant the vector space \mathbb{L} , but mixes this direction into the direct complement. Only two linearly

⁶For simplicity we don't pay any attention to the discrete structures in $O(1,3)$. The structures connected with the reflections in $O(1,3) \cong \mathbb{I}_2 \times_s (\mathbb{I}_2 \times SO^+(1,3))$ and the discrete symmetries in quantum field theories (T, CP, CPT) are treated elsewhere [8].

independent light-like vectors define a vector space decomposition [9], $M \cong \mathbb{L}_+ \oplus \mathbb{L}_- \oplus \mathbb{S}_2$ (Witt decomposition), out of which a space-time decomposition (Sylvester decomposition) can be constructed, $\mathbb{L}_+ \oplus \mathbb{L}_- \oplus \mathbb{S}_2 \cong \mathbb{T} \oplus \mathbb{S}_1 \oplus \mathbb{S}_2$. The little groups of this decomposition - in this case it is no fixgroup of a vector, but the stability group of the decomposition - are isomorphic to $SO(2) \cong U(1)$. The spectrum of this little group is discrete as can be seen by the polarisation states of the photons or, if massless, the helicity states of the neutrinos.

The 3rd class is the trivial one with little group being identical with the whole group $O(1, 3)$. There are no particles connected with this class.

The 4th class is characterized by imaginary mass. The little group of a space-like vector is isomorphic to the non-compact group $SO(1, 2)$ with continuous spectrum. Therewith this class is characterized by continuous spin, however, there are again no particles connected with this class⁷.

The manifold decompositions are decompositions of the strata with the pattern $D(1) \times SO^+(1, 3)/H$ with H denoting the little group, whereas the vector space decompositions, linear but local, recover the whole Minkowski space. Again this is consistent with our experience of space: We receive no direct knowledge from 3-space. All our information comes from within our current backward lightcone. 3-space is only a linear extrapolation, given together with the assumption, that the information comes from objects space-like to us in previous times⁸. I.e. the linear Minkowski space is only a linear reconstruction of space-time. Prior is the Lorentz orbit structure or an even more complicated Einstein space-time.

One remark to the classification of particles according to the Poincaré group: Due to the affine space-time translations within the Poincaré group particles are regarded to be asymptotic, i.e. far of from (point like?) ‘interactions’. The properties of the Poincaré group characterized by the Casimir invariants of the Poincaré group, mass and spin, are thus properties of the asymptotic particles. As can be seen in the standard model of elementary particles the ‘particles’ in the interaction are characterized by the Casimir

⁷Influenced by experiments on the spectrum of the beta decay of tritium, which gave negative mass squared for the neutrinos, some people speculated about neutrinos being tachyons [10]. If the Wigner classification due to the Poincaré group is correct, tachyons would have continuous spin, which for neutrinos contradicts the experimentally verified helicity structure. Only with the distinction of an additional vector, which breaks the little group $SO(1, 2)$ again down to $SO(2)$, there would arise particles with discrete spin.

⁸Maybe this is a rather natural assumption, however, it is an assumption.

invariants of only the Lorentz group or their covering group $SL(2, \mathcal{C})$, i.e. by chirality as expressed with left- and right-handed Weyl-spinors. These invariants are different to the Casimir invariants of the Poincaré group. Especially mass, connected with the translations, is no invariant of the Lorentz group and thus no good invariant of the interaction, but has to be renormalized⁹. The same seems to be true for charge. Again it is the Higgs sector in the standard model which connects the chirality structure of the Weyl spinors with the spin structure of the Dirac spinors and at the same time generates mass and charge. Therefore the Higgs mechanism may be seen as the link between the structures in the interaction and the asymptotics parametrized with particles.

Some Further Aspects of the Higgs Sector

The Higgs potential is used to motivate a non-trivial vector in the representation space $V \cong \mathcal{C}^2$ of an $U(2)$ -symmetry. It is the Higgs potential in which the introduction of a scale is hidden. Apart from the already mentioned symmetry structure the non-trivial Higgs vector defines locally, i.e. on the Goldstone manifold $U(2)/U(1)$, a vector space decomposition with one ‘direction’ (the ‘charge direction’) having the residual $U(1)$ -symmetry. This is parametrized with the Higgs field as follows:

The Higgs field $\phi(x) \in \mathcal{C}^2$ in its ‘symmetry breaking phase’ is linearly expanded from the non-trivial Higgs vector, e.g. $\phi_0 = \begin{pmatrix} 0 \\ v \end{pmatrix}$, in cartesian coordinates, regarded as infinitesimal excitations,

$$\phi(x) = \phi_0 + \begin{pmatrix} 0 \\ \eta(x) \end{pmatrix} + \begin{pmatrix} \phi_1(x) + i\phi_2(x) \\ i\phi_3(x) \end{pmatrix} \in \mathcal{C}^2. \quad (3)$$

The $\phi_i(x)$ are the Goldstone fields (excitations in directions of the ground state manifold) and the $\eta(x)$, the excitation in direction of the Higgs vector, becomes the massive Higgs particle. This is the linear expansion of \mathcal{C}^2 out of one non-trivial vector. It is according to the ‘tangent space decomposition’ $\mathcal{C}^2 \cong \mathbb{R} \oplus \mathbb{R}^3$. This tangent space decomposition is regarded to be done at the manifold decomposition of the stratum induced by the non-trivial Higgs vector: $\mathcal{C}^2 \cong D(1) \times U(2)/U(1) \uplus \{0\}$. The manifold decomposition is used

⁹Keep in mind, that the renormalization of mass can be seen as a representation of $D(1) \cong \mathbb{R}^+$.

for the Higgs field expansion in polar coordinates, adapted to the symmetry structure of the problem¹⁰:

$$\phi(x) = e^{\tilde{\eta}(x)} e^{i(\tilde{\phi}_i(x)\tau_i + \tilde{\phi}_0(x)\mathbf{1}_2)} \phi_0. \quad (4)$$

It is the dilatational excitation, $D(1) \notin U(2)$ but $D(1) \in GL(2, \mathcal{C})_{\mathbb{R}}$, which is connected with the massive Higgs particle¹¹.

Via the Yukawa couplings in the Weinberg-Salam model the properties of the Higgs sector are carried to the electron-neutrino vector space. I.e. one aspect of the Higgs sector is to define, within the $U(2)$ -representation space of the electron-neutrino¹², the ‘local’ difference between neutrino and electron, with residual non-trivial charge symmetry $U(1)$ for the electron. To be more precise: Symmetry means ‘there is no difference’ when acting with a certain operation and so there is no difference between electron and neutrino in what we call ‘interaction’. However, asymptotically we distinguish between electron and neutrino, both having different mass and charge. Therefore we have to distinguish between the electron-neutrino-field in the interaction and the electron and the neutrino in the asymptotics. This difference is defined by the non-trivial Higgs vector, leading to the concept of mass and charge for the particles.

Now at the latest the question arises, whether this needs to be caused by a potential or whether there may be another structure like in the Wigner classification. Let us face some more questions in this context: The Higgs mechanism introduces mass terms in the standard model Lagrange density in a gauge invariant way. On the other hand mass is defined as Casimir invariant of the Poincaré group. The later concept is used in the Wigner classification of particles, which we have seen can equally be interpreted in terms of symmetry breaking. Thus we have at least two concepts of mass: the ‘Higgs mass’ and the ‘Poincaré or Wigner mass’. These concepts of

¹⁰For simplicity we here used a representation of $U(2)$ rather than of $U(2)/U(1)$. Hence there is one ‘superfluous parameter’. E.g. for ϕ_0 given in a certain basis by $\begin{pmatrix} 0 \\ v \end{pmatrix}$ the action of $\mathbf{I}_2 + \tau_3$ is trivial, giving the residual charge- $U(1)$ representation. Therewith in this basis the combination $\phi_0(x) + \tilde{\phi}_3(x)$ is no parameter of the Goldstone manifold.

¹¹The groups $GL(2, \mathcal{C})$ and $SL(2, \mathcal{C})$ are regarded in the following as real Lie groups.

¹² $U(2)$ and $U(1) \times SU(2)$ have the same Lie algebra $u(1) \oplus su(2)$ because of the isomorphism $U(2) \cong (U(1) \times SU(2))/\mathbf{I}_2$. Due to the common discrete factor the representations of $U(2)$ are representations of $U(1) \times SU(2)$ with a specific correlation. The particle spectrum of the standard model shows exactly this correlation [5, 11]. Therefore we regard $U(2)$ to be the symmetry group of the electro-weak interaction.

mass are uncorrelated like it was the case for gravitational mass and inertial mass before general relativity. Could one believe that there are two different structures in special relativistic quantum field theory (omitting even general relativity with quite another aspect of mass) leading to the same concept of mass? Isn't it astonishing that both structures can be interpreted in terms of symmetry breaking? Shouldn't it be only one mechanism which generates the internal and external aspects of mass, i.e. only one symmetry breaking vector? Could it be that the $D(1)$ structures for massive particles in the Wigner case and in the Higgs case are quite the same, i.e. the massive vector in Minkowski space and the non-trivial vector for the Higgs mechanism are of the same origin? We will in the following section present a first attempt for a model based on the Higgs mechanism and on the space-time model of Saller [7] which will generate at least the symmetry structure of the Weinberg-Salam model and of a massive particle with only one symmetry breaking vector. The equality of 'Higgs-mass' and 'Wigner-mass' are thus evident per construction.

A Unified Model

Let us sum the structures of mass connected with symmetry breaking: On the one hand we have the external structure of massive objects decomposing the time-like stratum $D(1) \times SO^+(1, 3)/SO(3)$ with broken (or fixed) $D(1)$ scale invariance and residual symmetry $SO(3)$. On the other hand we have the internal structure decomposing \mathcal{C}^2 according to $D(1) \times U(2)/U(1) \uplus \{0\}$ again with fixed $D(1)$ scale but residual symmetry $U(1)$. How do these structures fit together by identifying the dilatation structures? Notice the observation in [7]:

$$SO^+(1, 3)/SO(3) \cong SL(2, \mathcal{C})/SU(2) \cong UL(2, \mathcal{C})/U(2), \quad (5)$$

with $UL(2, \mathcal{C}) := \{g \in GL(2, \mathcal{C}) \mid |\det g| = 1\} \cong GL(2, \mathcal{C})/D(1)$ the group of linear operations with determinant of modulus one¹³. Hence we should analyze the action of the whole $GL(2, \mathcal{C})$.

Imagine the action of $GL(2, \mathcal{C})$ on any non-trivial vector v in $V \cong \mathcal{C}^2$: There is no natural basis in V . Thus this vector defines one (normal) basis

¹³This group will be named according to [7] unimodular linear group. In general $UL(V) := \{g \in GL(V) \mid |\det g| = 1\}$. For every finite dimensional vector space V we have the direct group product separation: $GL(V) \cong D(1) \times UL(V)$.

vector and all other vectors in V can be ‘measured’ with respect to v if there is in addition a bi- or sesquilinear form. Hence together with a positive definite sesquilinear form (which is equivalent to a positive definite and therewith $U(2)$ -invariant conjugation on \mathcal{C}^2) it defines a ‘length’ or ‘energy’ scale in V . In $GL(2, \mathcal{C})$ the dilatations are separated naturally according to the direct group product $GL(2, \mathcal{C}) \cong D(1) \times UL(2, \mathcal{C})$. However, the action of $GL(2, \mathcal{C})$ on a non-trivial vector $v \in \mathcal{C}^2$ together with a positive definite conjugation decomposes this group according to

$$GL(2, \mathcal{C}) \bullet v \cong (D(1) \times UL(2, \mathcal{C})/U(2) \times U(2)/U(1) \times U(1)) \bullet v. \quad (6)$$

We have to emphasize that this is a manifold decomposition, no direct group product decomposition. The later part of this decomposition has the symmetry structure of the Higgs sector in the Weinberg-Salam model: $U(1)$ is the little group of the non-trivial vector v acting trivial. The action of $U(2)/U(1)$, being the Goldstone manifold $[v]_{U(2)}$, generates vectors in \mathcal{C}^2 which have the same ‘length’¹⁴ and thus introducing the same length or energy scale. The fixgroups of the elements of this coset space are isomorphic but not identic. I.e. $U(1)$ is local on the Goldstone manifold. The action of the coset

$$UL(2, \mathcal{C})/U(2) \cong SL(2, \mathcal{C})/SU(2) \cong SO^+(1, 3)/SO(3)$$

on the non-trivial vectors in $[v]_{U(2)}$ is trivial. Regarded as manifold it is isomorphic to the Lorentz orbit of a massive particle. I.e. a Lorentz boost doesn’t change the (energy) scale, but defines equivalent space-time decompositions. Moreover, $D(1) \times UL(2, \mathcal{C})/U(2)$ is isomorphic to the stratum of massive particles, the interior part of the (forward) lightcone, with tangent space being the Minkowski space [7] in its local space-time decompositions. Hence, we have the non-linear space-time structure for a massive object on the one side [7],

$$D(1) \times UL(2, \mathcal{C})/U(2) \cong GL(2, \mathcal{C})/U(2),$$

¹⁴We used a positive definite conjugation to define the group $U(2)$ within $GL(2, \mathcal{C})$ and therewith a positive definite sesquilinear form providing the concept of a positive definite (length or energy) scale. We could have equally used the indefinite conjugation on \mathcal{C}^2 with $U(1, 1)$ invariance group. This would also introduce a sesquilinear form for measuring vectors in \mathcal{C}^2 with respect to v , but with indefinite results. I.e. it would introduce a scale, but not interpretable in general as energy scale. The decomposition would be according to $GL(2, \mathcal{C}) \cong D(1) \times UL(2, \mathcal{C})/U(1, 1) \times U(1, 1)/U(1) \times U(1)$.

and the symmetry structure of the Higgs sector in the standard model on the other side [3],

$$U(2)/U(1) \times U(1) \cong U(2),$$

giving together the action of $GL(2, \mathcal{C})$:

$$GL(2, \mathcal{C}) \cong \left(\underbrace{D(1) \times UL(2, \mathcal{C})/U(2)}_{\text{space-time}} \times \underbrace{U(2)/U(1) \times U(1)}_{\text{Higgs sector}} \right)$$

Notice that the action of $U(2)$ again is local on the space-time manifold $GL(2, \mathcal{C})/U(2)$, i.e. the action of $U(2)$ is space-time dependent.

The decomposition of $GL(2, \mathcal{C})$ according to (6) is no group product decomposition. I.e. the group $GL(2, \mathcal{C})$ itself acts not irreducibly on this decomposition. Only subgroups of $GL(2, \mathcal{C})$ have the possibility to act irreducible on parts of this decomposition. Hence the action of $GL(2, \mathcal{C})$ on the manifold decomposition (6) will be quite non-trivial, resulting in a new isomorphic, but different decomposition.

What makes this simple model of the action of $GL(2, \mathcal{C})$ on a non-trivial vector worth being considered? First of all, one has only one symmetry breaking mechanism generating the internal and external symmetry structure for massive particles and introducing only one mass or energy scale. Higgs-mass and Wigner-mass are identical a priori. Moreover, the generation of external and internal symmetry structures (besides the symmetry structure for quarks) ‘at the same time’ may lead to the occurrence of space-time and matter (particles, objects) on equal level: Since Newton we organize objects and their motion in our space-time, regarded space-time being a priori. This is like distributing objects in a given box. General relativity relates the geometry of this box to the distribution of its contents, the box itself (the Einstein manifold) stays a priori. This results in the difference of geometry on the one side and matter on the other side in the Einstein equation, a duality which Einstein himself always tried to resolve. But how do we experience space-time? According to Mach’s principle one can only define energy or mass, momentum, and angular momentum with respect to other objects. The same is true for length or volume. Even more, we can test space-time itself only with the help of its ‘contents’. According to the philosophy of Leibniz and Mach space-time can only be regarded as relation between objects, i.e. space-time without contents doesn’t exist at all. But what was first, space-time or matter? This is like the problem of the hen and the egg. The solution of this paradox is, that none ‘was first’, but both structures are

generated ‘at the same time’, i.e. space-time only comes along with matter and matter only comes along with space-time. The above model generates external and internal symmetries out of one root and therewith has the potential to generate space-time and matter ‘at the same time’. This is done with the help of the concept of symmetry breaking given in a more general framework.

Outlook

In the context of the structures suggested above there immediately arise some questions. Let us mention some of these questions and hints to adress them instead of a conclusion:

Where does the non-trivial vector v come from when not spontaneously chosen as ground state of a potential? In quantized theories it is not the Higgs vector which is non-trivial, but the vacuum expectation value of the Higgs field. Hence, when there seems to be a symmetry breaking ground state it could be the ‘vacuum’ of a Fock space in a quantized system out of which the whole Fock space will be generated. The calculation of the spectrum of a quantized theory of \mathcal{C}^2 with non-invariant Fock ground state is possible together with the canonical quantization of a vector space $V \cong \mathcal{C}^n$ in the basis-free formulation according to H. Saller [12]. The quantization of \mathcal{C} gives just the quantum mechanical harmonic oscillator with its spectrum. The quantization of \mathcal{C}^2 with non-trivial Fock ground state will generate a spectrum of ‘particles’ which at least will show the symmetry structure of (6).

On the other hand we up to now only dealt with one symmetry breaking vector. Which structures will arise when there are two different vectors with isomorphic, but different separations according to (6)? Is there the possibility of an even more complicated Einstein manifold instead of the forward time-like cone structure, generated out of more than one non-trivial vector? The example of the earth hints that there is the possibility of ‘Coriolis forces’ when breaking the symmetry in different manners. Could gravity be the result of generalized Coriolis forces caused by two or more different symmetry breaking structures? dynamical,

Two different vectors in \mathcal{C}^2 are connected by the action of a distinct element $g \in GL(2, \mathcal{C})$. But how does the action of $g \in GL(2, \mathcal{C})$ change the manifold decomposition (6)? And how would we interpret this isomorphic, but different decomposition and therewith the action of $GL(2, \mathcal{C})$ itself?

If we generate the concept of mass and energy out of deeper structures via symmetry breaking, what is the concept of dynamics and therewith how do thus equation of motions (strongly correlated to the concept of time) or Lagrange densities (strongly correlated to the concept of energy) evolve?

We only considered purely structural concepts. Which aspects will arise when making this system dynamical? Does there appear Goldstone modes together with the two cosets $U(2)/U(1)$ and $UL(2, \mathcal{C})/U(2) \cong SO^+(1, 3)/SO(3)$ and what will be their role especially in the case of the non-compact coset? Is there a connection to the geometry on this coset space?

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